

ON SOME TWISTED COHOMOLOGY OF THE RING OF INTEGERS

SEOK-MIN LEE*

ABSTRACT. As an analogy of Poincaré series in the space of modular forms, T. Ono associated a module M_c/P_c for $\gamma = [c] \in H^1(G, R^\times)$ where finite group G is acting on a ring R . M_c/P_c is regarded as the 0-dimensional twisted Tate cohomology $\hat{H}^0(G, R^+)_\gamma$. In the case that G is the Galois group of a Galois extension K of a number field k and R is the ring of integers of K , the vanishing properties of M_c/P_c are related to the ramification of K/k . We generalize this to arbitrary n -dimensional twisted cohomology of the ring of integers and obtain the extended version of theorems. Moreover, some explicit examples on quadratic and biquadratic number fields are given.

1. Introduction

T. Ono developed an algebraic analogy of the space of modular forms and the subspace of Poincaré series in the cohomological point of view, as a form of quotient module. Let G be a finite group acting on a ring R with the unit 1 and denote the action by $(s, a) \mapsto {}^s a$, $a \in R$, $s \in G$. Since the group G acts on the unit group R^\times of R , we can consider the 1-cocycle set

$$Z^1(G, R^\times) = \{c : G \rightarrow R^\times; c_{st} = c_s {}^s c_t, s, t \in G\},$$

where $c_s = c(s)$ for a cocycle c and $s \in G$. Two cocycles c, c' are equivalent (cohomologous), denoted by $c \sim c'$, if there is a unit element $u \in R^\times$ such that $c'_s = u^{-1} c_s {}^s u$, $s \in G$. The cohomology set is defined by

$$H^1(G, R^\times) = Z^1(G, R^\times) / \sim.$$

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For a 1-cocycle $c \in Z^1(G, R^\times)$, T. Ono defined the modules M_c and P_c , named after modular form and Poincaré series, given by

$$M_c = \{a \in R; c_s {}^s a = a, s \in G\}, \text{ and}$$

$$P_c = \{p_c(a) = \sum_{t \in G} c_t {}^t a; a \in R\},$$

where $p_c(a)$ is called the *Poincaré sum* [4, 5, 6]. The module structure of M_c/P_c depends only on the cohomology class $\gamma = [c] \in H^1(G, R^\times)$ containing c . In particular, if $c \sim 1$, the module M_c/P_c is equal to the 0-dimensional Tate cohomology $\widehat{H}^0(G, R)$. For a general cocycle c , M_c/P_c can be considered as the twisted 0-Tate cohomology $\widehat{H}^0(G, R)_\gamma$ with the new action $a \mapsto c_s {}^s a$. The case that G is the Galois group of a Galois extension K/k of number fields and R is the ring of integers \mathcal{O}_K of K was studied in [4, 5, 6, 7]. T. Ono proved in [7] the following theorems;

THEOREM 1.1. *Let K/k be a finite Galois extension of number fields. If K/k is unramified or tamely ramified, then $M_c = P_c$ for all cocycle $c \in Z^1(\text{Gal}(K/k), \mathcal{O}_K^\times)$.*

THEOREM 1.2. *Let K/k be a finite Galois extension of number fields and $G = \text{Gal}(K/k)$. For a cocycle $c \in Z^1(G, \mathcal{O}_K^\times)$ denote by $c_{\mathfrak{p}}$ the cocycle induced from c by localization at \mathfrak{p} . Then we have the product relation $(M_c : P_c) = \prod_{\mathfrak{p}} (M_{c_{\mathfrak{p}}} : P_{c_{\mathfrak{p}}})$ where for each \mathfrak{p} we choose one \mathfrak{P} dividing \mathfrak{p} .*

In this work, these theorems are generalized as follows: First, we define n -dimensional twisted cohomology $H^n(G, \mathcal{O}_K)_c$ in section 2, then we show that if K/k is tamely ramified then the twisted n -cohomology vanishes for any integer n and for any cocycle c (Theorem 3.1), and that

$$H^n(G, \mathcal{O}_K)_c \cong \prod_{\mathfrak{p}} H^n(G_{\mathfrak{p}}, \mathcal{O}_{K_{\mathfrak{p}}})_{c_{\mathfrak{p}}},$$

where for each \mathfrak{p} we choose one \mathfrak{P} dividing \mathfrak{p} (Theorem 5.1).

On the other hand, Yokoi [9, 10, 11] and Lee and Madan [3] studied n -dimensional Galois cohomology of the ring of integers. We may consider this paper as a generalization of their works [3] and [9, 10, 11] into the twisted cohomology as well.

Moreover, in Theorem 3.2 it is shown that for the cyclic Galois extension, we have $|H^n(G, \mathcal{O}_K)_c| = |M_c/P_c|$ for all positive integer n and for all 1-cocycle $c \in Z^1(G, \mathcal{O}_K^\times)$. This does not hold for a non-cyclic extension, e.g. biquadratic fields (Section 7).

2. n -dimensional twisted cohomology

For a 1-cocycle $c \in Z^1(G, R^\times)$, we define the twisted action $(s, a) \mapsto c_s {}^s a$ of G on the additive group R . We can consider R as the G -module defined by the twisted action and let us call it a c -twisted G -module. If A is a G -module derived from the additive group R which admits the twisted action by c , then also we can consider A as a c -twisted G -module.

For a c -twisted G -module A , we have n -dimensional twisted cohomology group $H^n(G, R)_c$ for each nonnegative integer n as follows: Let $C^n(G, A)_c$, the set of n -chains, be the set of all maps of G^n to A for $n > 0$ and let $C^0(G, A)_c = \{1_G\}$. The coboundary map $d_{n+1} : C^n(G, A)_c \rightarrow C^{n+1}(G, A)_c$ is defined by

$$\begin{aligned} (d_{n+1}f)(s_1, \dots, s_{n+1}) &= c_{s_1} {}^{s_1} f(s_2, \dots, s_{n+1}) \\ &+ \sum_{i=1}^n (-1)^i f(s_1, \dots, s_i s_{i+1}, \dots, s_{n+1}) \\ &+ (-1)^{n+1} f(s_1, \dots, s_n) \end{aligned}$$

then $\text{im } d_n \subset \ker d_{n+1}$. Denote by $H^n(G, R)_c = Z^n(G, R)_c / B^n(G, R)_c$ where $Z^n(G, R)_c = \ker d_{n+1}$ is the the twisted n -cocycle and $B^n(G, R)_c = \text{im } d_n$ is the twisted n -coboundary. Here we have $H^0(G, R)_c = M_c$ and $\widehat{H}^0(G, R)_c = M_c / P_c$. The twisted 1-cocycle in A is the function $d : G \rightarrow A$ such that

$$d_{st} = d_s + c_s {}^s d_t$$

and the twisted 1-coboundary is $d_s = c_s {}^s b - b$ for some $b \in A$.

A homomorphism between c -twisted G -modules is well defined, and induces a group homomorphism between cohomologies. As usual, if

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is an exact sequence of c -twisted G -modules, then we have the long exact sequence of twisted cohomology groups:

$$\begin{aligned} 0 \rightarrow H^0(G, A)_c &\rightarrow H^0(G, B)_c \rightarrow H^0(G, C)_c \rightarrow H^1(G, A)_c \\ &\rightarrow H^1(G, B)_c \rightarrow H^1(G, C)_c \rightarrow H^2(G, A)_c \rightarrow H^2(G, B)_c \\ &\rightarrow H^2(G, C)_c \rightarrow \dots \end{aligned}$$

Now assume that the group G is cyclic with the generator s of order n . Define two endomorphisms Δ and N of c -twisted G -module A such

that

$$\Delta = 1 - c_s s, \quad N = \sum_{i=0}^{n-1} c_s^i s^i.$$

Note that $c_s^i = c_s {}^s c_s {}^{s^2} c_s \cdots {}^{s^{i-1}} c_s$. Then we have $\Delta N = N \Delta = 0$ and this means $\text{im } N \subset \ker \Delta$ and $\text{im } \Delta \subset \ker N$. It turns out to be

$$\widehat{H}^0(G, A)_c = \ker \Delta / \text{im } N, \quad H^1(G, A)_c = \ker N / \text{im } \Delta$$

and Herbrand theorem holds: If

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is an exact sequence of c -twisted G -modules, then the twisted cohomology exact sequence is an exact hexagon:

$$\begin{array}{ccccc} & & \widehat{H}^0(G, A)_c & \longrightarrow & \widehat{H}^0(G, B)_c \\ & \nearrow & & & \searrow \\ H^1(G, C)_c & & & & \widehat{H}^0(G, C)_c \\ & \nwarrow & & & \swarrow \\ & & H^1(G, B)_c & \longleftarrow & H^1(G, A)_c \end{array}$$

We denote by $h_c(A) = |\widehat{H}^0(G, A)_c|/|H^1(G, A)_c|$ the Herbrand quotient of the twisted cohomology. If two of three Herbrand quotients $h_c(A)$, $h_c(B)$, $h_c(C)$ are defined, so is the third and we have

$$h_c(B) = h_c(A)h_c(C).$$

Now we will verify that for $n \geq 1$, the structure of the twisted cohomology $H^n(G, R)_c$ twisted by c depends only on the cohomology class $\gamma = [c] \in H^1(G, R^\times)$. Let $c \sim c'$, i.e. for some $u \in R^\times$,

$$c'_s = u^{-1} c_s {}^s u \quad \text{for all } s \in G.$$

Then

$$\begin{aligned}
f \in Z^n(G, R)_{c'} &\Leftrightarrow f : G^n \rightarrow R \text{ such that , for } s_1, \dots, s_{n+1} \in G, \\
& c'_{s_1} s_1 f(s_2, \dots, s_{n+1}) \\
&= \sum_{i=1}^n (-1)^{i+1} f(s_1, \dots, s_i s_{i+1}, \dots, s_{n+1}) \\
&+ (-1)^n f(s_1, \dots, s_n) \\
&\Leftrightarrow c_{s_1} s_1 (uf(s_2, \dots, s_{n+1})) \\
&= \sum_{i=1}^n (-1)^{i+1} uf(s_1, \dots, s_i s_{i+1}, \dots, s_{n+1}) \\
&+ (-1)^n uf(s_1, \dots, s_n) \\
&\Leftrightarrow uf \in Z^n(G, R)_c.
\end{aligned}$$

And

$$\begin{aligned}
g \in B^n(G, R)_{c'} &\Leftrightarrow g : G^n \rightarrow R \text{ such that there exists } f : G^{n-1} \rightarrow R, \\
& g(s_1, \dots, s_n) = c'_{s_1} s_1 f(s_2, \dots, s_n) \\
&+ \sum_{i=1}^{n-1} (-1)^i f(s_1, \dots, s_i s_{i+1}, \dots, s_n) \\
&+ (-1)^{n+1} f(s_1, \dots, s_{n-1}) \\
&\Leftrightarrow ug(s_1, \dots, s_n) = c_{s_1} s_1 (uf(s_2, \dots, s_n)) \\
&+ \sum_{i=1}^{n-1} (-1)^i uf(s_1, \dots, s_i s_{i+1}, \dots, s_n) \\
&+ (-1)^{n+1} uf(s_1, \dots, s_{n-1}) \\
&\Leftrightarrow ug \in B^n(G, R)_c.
\end{aligned}$$

Hence we have

$$\begin{aligned}
H^n(G, R)_c &= Z^n(G, R)_c / B^n(G, R)_c \\
&= uZ^n(G, R)_{c'} / uB^n(G, R)_{c'} \cong H^n(G, R)_{c'}.
\end{aligned}$$

We state this result in the following proposition.

PROPOSITION 2.1. *If c and c' are cohomologous in $Z^1(G, \mathcal{O}_K^\times)$, then we have*

$$H^n(G, R)_c \cong H^n(G, R)_{c'}.$$

3. Galois extension of number fields

Let K/k be a finite Galois extension of number fields. Through this paper, we will take the group G as the Galois group $\text{Gal}(K/k)$ of the extension and the ring R as the ring of integers \mathcal{O}_K of K . However, firstly we consider the case $R = K$. By Hilbert's Theorem 90, any cocycle $c \in Z^1(G, K^\times)$ is a coboundary, that implies, $H^n(G, K)_c \cong H^n(G, K)$, which vanishes for all dimension n . Now we take $R = \mathcal{O}_K$. A cocycle $c \in Z^1(G, \mathcal{O}_K^\times)$ can be regarded as a cocycle into K^\times , then \mathcal{O}_K can be dealt as a twisted submodule of K .

We have the generalization of Theorem 1.1 on n -dimensional twisted cohomology.

THEOREM 3.1. *If the finite Galois extension K/k of number fields is tamely ramified, then $H^n(G, \mathcal{O}_K)_c$ vanishes for all dimension $n \geq 1$ and for all cocycle $c \in Z^1(G, \mathcal{O}_K^\times)$.*

Proof. Let F be an intermediate field of K/k and Γ be the Galois group of K over F . Since Γ is a subgroup of G , any c -twisted G -module is c -twisted Γ -module, and twisted cohomology of Γ makes sense. We will verify that for any intermediate field F of K/k with the Galois group $\Gamma = \text{Gal}(K/F)$, the 0- and 1-dimensional twisted Tate cohomology groups vanish. Then the result follows.

- (1) For $c \in Z^1(G, \mathcal{O}_K^\times)$, let c' be the restriction of c on Γ then $c' \in Z^1(\Gamma, \mathcal{O}_K^\times)$. Note that $\widehat{H}^0(\Gamma, \mathcal{O}_K)_c = \widehat{H}^0(\Gamma, \mathcal{O}_K)_{c'}$. Since K/k is tamely ramified, K/F is also tamely ramified. By Theorem 1.1 and the above assertion, we have $\widehat{H}^0(\Gamma, \mathcal{O}_K)_c = 0$.
- (2) Since K/F is tamely ramified, the trace map $\text{Tr}_{K/F} : \mathcal{O}_K \rightarrow \mathcal{O}_F$ is surjective so there exists an element $\alpha \in \mathcal{O}_K$ such that $\text{Tr}_{K/F}(\alpha) = 1$. Let $c \in Z^1(G, \mathcal{O}_K^\times)$ be a 1-cocycle. For any twisted 1-cocycle $d \in Z^1(\Gamma, \mathcal{O}_K)_c$, define $\beta = \sum_{t \in \Gamma} d_t {}^t\alpha \in \mathcal{O}_K$ the Poincaré sum for the twisted cocycle d then we have

$$\begin{aligned}
 \beta - c_s {}^s\beta &= \sum_{t \in \Gamma} d_t {}^t\alpha - \sum_{t \in \Gamma} c_s {}^s d_t {}^{st}\alpha \\
 (3.1) \quad &= \sum_{t \in \Gamma} d_t {}^t\alpha - \sum_{t \in \Gamma} (d_{st} - d_s) {}^{st}\alpha \\
 &= \sum_{t \in \Gamma} d_t {}^t\alpha - \sum_{t \in \Gamma} d_{st} {}^{st}\alpha + d_s \sum_{t \in \Gamma} {}^{st}\alpha \\
 &= d_s \cdot \text{Tr}_{K/F}(\alpha) = d_s.
 \end{aligned}$$

This implies that any twisted 1-cocycle is a twisted 1-coboundary. \square

THEOREM 3.2. *Let K/k be a finite cyclic Galois extension of number fields with the Galois group $G = \text{Gal}(K/k)$, and let $c \in Z^1(G, \mathcal{O}_K^\times)$ be a 1-cocycle of G in the unit group of \mathcal{O}_K . Then the twisted cohomology group $H^n(G, \mathcal{O}_K)_c$ has the same order with $\widehat{H}^0(G, \mathcal{O}_K)_c = M_c/P_c$ for every dimension $n \geq 1$.*

Proof. Let s be the generator of the group G of order m . We see M_c as a twisted G -module. Since M_c makes an ideal of \mathcal{O}_k by multiplied by a proper element of \mathcal{O}_K as in [7], the rank of M_c is same as \mathcal{O}_k . On the other hand, let $\beta \in \mathcal{O}_K$ be the element generating the normal basis for K/k . Then $\{c_{s^1} s^1 \beta, c_{s^2} s^2 \beta, \dots, c_{s^m} s^m \beta\}$ is also a basis of K/k since the basis change matrix is diagonal with entries c_{s^i} , the unit elements. Define

$$M = M_c c_{s^1} s^1 \beta + M_c c_{s^2} s^2 \beta + \dots + M_c c_{s^m} s^m \beta$$

then we have that the index $[\mathcal{O}_K : M]$ is finite and that M is G -regular as a twisted G -module since each $M_c c_{s^i} s^i \beta$ forms a direct summand of M . Let $N = \mathcal{O}_K/M$ then N is a finite twisted G -module. We have an exact sequence of twisted modules:

$$0 \rightarrow M \rightarrow \mathcal{O}_K \rightarrow N \rightarrow 0$$

Since G is cyclic, we have $H^{2l}(G, \mathcal{O}_K)_c \cong \widehat{H}^0(G, \mathcal{O}_K)_c$ and $H^{2l-1}(G, \mathcal{O}_K)_c \cong H^1(G, \mathcal{O}_K)_c$ for any positive integer l . $H^n(G, M)_c$ vanishes for any n because M is a twisted G -regular module. Hence we have $\widehat{H}^0(G, \mathcal{O}_K)_c \cong \widehat{H}^0(G, N)_c$ and $H^1(G, \mathcal{O}_K)_c \cong H^1(G, N)_c$. Also, since N is a finite module, we have that the Herbrand quotient $h_c(N)$ of the twisted cohomology of N is equal to 1, which concludes the theorem. \square

4. 1-dimensional twisted cohomology

In this section, we study the structure of 1-dimensional twisted cohomology. We give an analogy of the group $H_{K/k}(\mathfrak{a})$ suggested in [10] into the case of twisted cohomology. Let a 1-cocycle $c \in Z^1(G, \mathcal{O}_K^\times)$ and a nonzero integer $a \in \text{Tr}_{K/k}(\mathcal{O}_K)$ in k be fixed. For a twisted 1-cocycle $d \in Z^1(G, \mathcal{O}_K)_c$, we can choose an integer β in \mathcal{O}_K such that

$$(4.1) \quad \beta - c_s s \beta = a \cdot d_s \quad \text{for all } s \in G.$$

Indeed, let α be the element in \mathcal{O}_K with the trace value a and let $\beta = \sum_{t \in G} d_t {}^t \alpha \in \mathcal{O}_K$ be of form of the Poincaré sum for the twisted cocycle d then we get the equation (4.1) with the similar assertion to (3.1). From the equation (4.1) we have that

$$(4.2) \quad \beta \equiv c_s {}^s \beta \pmod{a\mathcal{O}_K}.$$

Conversely, if $\beta \in \mathcal{O}_K$ satisfies the congruence relation (4.2) then β defines a twisted 1-cocycle $d \in Z^1(G, \mathcal{O}_K)_c$. Indeed, for each $s \in G$, let $d_s \in \mathcal{O}_K$ such that $\beta - c_s {}^s \beta = a \cdot d_s$ then d_s satisfies the condition of twisted 1-cocycle $d_{st} = d_s + c_s {}^s d_t$. Now we obtain the following proposition.

PROPOSITION 4.1. *Denote by*

$$Z_c(a) = \{\beta \in \mathcal{O}_K ; \beta \equiv c_s {}^s \beta \pmod{a\mathcal{O}_K}\}$$

then we have

$$Z^1(G, \mathcal{O}_K)_c \cong Z_c(a)/M_c.$$

Proof. Define the map $\varphi : Z_c(a) \rightarrow Z^1(G, \mathcal{O}_K)_c$ as above: $\beta \mapsto d$ such that $\beta - c_s {}^s \beta = a \cdot d_s$. Then φ is surjective by the above assertion and trivially φ is a homomorphism. One sees that $\ker \varphi = M_c$ easily. \square

Now, for the twisted 1-coboundary, we have

PROPOSITION 4.2. *Denote by*

$$B_c(a) = \{\beta \in \mathcal{O}_K ; \beta \equiv b \pmod{a\mathcal{O}_K} \text{ for some } b \in M_c\}$$

then we have

$$B^1(G, \mathcal{O}_K)_c \cong B_c(a)/M_c.$$

Proof. First note that clearly $B_c(a) \subset Z_c(a)$. The homomorphism $\varphi : Z_c(a) \rightarrow Z^1(G, \mathcal{O}_K)_c$ in the Proposition 4.1 induces an epimorphism $\varphi : B_c(a) \rightarrow B^1(G, \mathcal{O}_K)_c$ since

$$\begin{aligned} \beta \in B_c(a) &\Leftrightarrow \beta \equiv b \pmod{a\mathcal{O}_K} \text{ for some } b \in M_c \\ &\Leftrightarrow \beta - a\gamma \in M_c \text{ for some } \gamma \in \mathcal{O}_K \\ &\Leftrightarrow \beta - a\gamma = c_s {}^s (\beta - a\gamma) \text{ for some } \gamma \in \mathcal{O}_K \\ &\Leftrightarrow \beta - c_s {}^s \beta = a(\gamma - c_s {}^s \gamma) \text{ for some } \gamma \in \mathcal{O}_K \end{aligned}$$

and $\beta \mapsto \gamma - c_s {}^s \gamma \in B^1(G, \mathcal{O}_K)_c$. It is easy to see that the kernel is also M_c . \square

As a corollary of Proposition 4.1 and Proposition 4.2, we have

PROPOSITION 4.3. *Let a be a nonzero element of $Tr_{K/k}(\mathcal{O}_K)$. Then the 1-dimensional twisted Galois cohomology group $H^1(G, \mathcal{O}_K)_c$ of \mathcal{O}_K is isomorphic to the factor group*

$$H^1(G, \mathcal{O}_K)_c \cong Z_c(a)/B_c(a).$$

Now we define for an ideal \mathfrak{a} of \mathcal{O}_k ,

$$Z_c(\mathfrak{a}) := \{\alpha \in \mathcal{O}_K ; \alpha \equiv c_s {}^s\alpha \pmod{\mathfrak{a}\mathcal{O}_K}\}$$

$$B_c(\mathfrak{a}) := \{\alpha \in \mathcal{O}_K ; \alpha \equiv b \pmod{\mathfrak{a}\mathcal{O}_K} \text{ for some } b \in M_c\} = M_c + \mathfrak{a}\mathcal{O}_K$$

and

$$H_c(\mathfrak{a}) := Z_c(\mathfrak{a})/B_c(\mathfrak{a}).$$

If $\alpha \in Z_c(\mathfrak{a})$ then $p_c(\alpha) = \sum_{s \in G} c_s {}^s\alpha \equiv |G|\alpha \pmod{\mathfrak{a}\mathcal{O}_K}$ so that we have $|G|Z_c(\mathfrak{a}) \subset B_c(\mathfrak{a}) \subset Z_c(\mathfrak{a})$. In the case of principal ideal (a) of k , let us denote $H_c(a) = H_c((a))$ just for convenience. Proposition 4.3 shows that $H^1(G, \mathcal{O}_K)_c \cong H_c(a)$ for any trace image $a \in Tr_{K/k}(\mathcal{O}_K)$.

PROPOSITION 4.4. *Let \mathfrak{a} and \mathfrak{b} be mutually prime ideals of \mathcal{O}_k . Then we have*

$$H_c(\mathfrak{a}\mathfrak{b}) \cong H_c(\mathfrak{a}) \oplus H_c(\mathfrak{b}).$$

Proof. Let $\gamma \in Z_c(\mathfrak{a}\mathfrak{b})$. Since $\gamma - c_s {}^s\gamma \in \mathfrak{a}\mathfrak{b}\mathcal{O}_k \subset \mathfrak{a}\mathcal{O}_k \cap \mathfrak{b}\mathcal{O}_k$, we have $\gamma \in Z_c(\mathfrak{a})$ and $\gamma \in Z_c(\mathfrak{b})$ so that the natural homomorphism

$$Z_c(\mathfrak{a}\mathfrak{b}) \longrightarrow Z_c(\mathfrak{a})/B_c(\mathfrak{a}) \oplus Z_c(\mathfrak{b})/B_c(\mathfrak{b})$$

is induced. The kernel of the homomorphism is $B_c(\mathfrak{a}\mathfrak{b})$. Indeed, let $\gamma \in Z_c(\mathfrak{a}\mathfrak{b})$ such that $\gamma \in B_c(\mathfrak{a}) \cap B_c(\mathfrak{b})$, say, $\gamma - a_0 = \alpha \in \mathfrak{a}\mathcal{O}_K$ and $\gamma - b_0 = \beta \in \mathfrak{b}\mathcal{O}_K$ for some $a_0, b_0 \in M_c$. Since \mathfrak{a} and \mathfrak{b} are relatively prime, there are elements a and b in \mathfrak{a} and \mathfrak{b} , respectively, such that $a + b = 1$. Then $ab_0 + ba_0 \in M_c$ and

$$\gamma - (ab_0 + ba_0) = (a + b)\gamma - (ab_0 + ba_0) = a\beta + b\alpha \in \mathfrak{a}\mathfrak{b}\mathcal{O}_K$$

which means $\gamma \in B_c(\mathfrak{a}\mathfrak{b})$.

Now, it remains to show that the homomorphism is surjective: let $\alpha \in Z_c(\mathfrak{a})$ and $\beta \in Z_c(\mathfrak{b})$. Define $\gamma = a\beta + b\alpha$ where $a \in \mathfrak{a}$, $b \in \mathfrak{b}$ are fixed, with the property $a + b = 1$. Then we have $\gamma - c_s {}^s\gamma = a(\beta - c_s {}^s\beta) + b(\alpha - c_s {}^s\alpha) \in \mathfrak{a}\mathfrak{b}\mathcal{O}_K$ so that $\gamma \in Z_c(\mathfrak{a}\mathfrak{b})$. Since $a \in B_c(\mathfrak{a}) = M_c + \mathfrak{a}\mathcal{O}_K$ and $b \in B_c(\mathfrak{b}) = M_c + \mathfrak{b}\mathcal{O}_K$, we have $\gamma = a\beta + b\alpha \equiv \alpha \pmod{B_c(\mathfrak{a})}$ and $\gamma \equiv \beta \pmod{B_c(\mathfrak{b})}$. \square

PROPOSITION 4.5. *For any ideal \mathfrak{a} of \mathcal{O}_k , the group $H_c(\mathfrak{a})$ is finite. More precisely, we have*

$$[B_c(\mathfrak{a}) : \mathfrak{a}\mathcal{O}_K] = N(\mathfrak{a}),$$

and the order of $H_c(\mathfrak{a})$ divides $N(\mathfrak{a})^{n-1}$ where n is the degree of the extension K/k . In particular, for a prime ideal \mathfrak{p} of \mathcal{O}_k , the order $|H_c(\mathfrak{p}^m)|$ is a power of a prime integer p (possibly $1 = p^0$) where $\mathfrak{p} \mid p$.

Proof. For any ideal \mathfrak{a} of \mathcal{O}_k , we have $\mathfrak{a}\mathcal{O}_K \subset B_c(\mathfrak{a}) \subset Z_c(\mathfrak{a}) \subset \mathcal{O}_K$ as additive groups. So we have that $|H_c(\mathfrak{a})| = [Z_c(\mathfrak{a}) : B_c(\mathfrak{a})]$ is finite. On the other hand, it is known that there exists a nonzero integer ξ in \mathcal{O}_K such that $\xi M_c = \mathfrak{b} = \xi\mathcal{O}_K \cap \mathcal{O}_k$ is an ideal of \mathcal{O}_k , and $\xi\mathcal{O}_K$ is an ambiguous ideal of \mathcal{O}_K . Then $\xi\mathfrak{a}\mathcal{O}_K \cap \mathcal{O}_k = \mathfrak{a}\mathfrak{b}$ and we have

$$\frac{B_c(\mathfrak{a})}{\mathfrak{a}\mathcal{O}_K} \cong \frac{\mathfrak{b} + \xi\mathfrak{a}\mathcal{O}_K}{\xi\mathfrak{a}\mathcal{O}_K} \cong \frac{\mathfrak{b}}{\mathfrak{b} \cap \xi\mathfrak{a}\mathcal{O}_K} = \frac{\mathfrak{b}}{\mathfrak{a}\mathfrak{b}}.$$

Hence $[B_c(\mathfrak{a}) : \mathfrak{a}\mathcal{O}_K] = N(\mathfrak{a})$. We obtain that $|H_c(\mathfrak{a})| = [Z_c(\mathfrak{a}) : B_c(\mathfrak{a})]$ is a divisor of $[\mathcal{O}_K : B_c(\mathfrak{a})] = N(\mathfrak{a}\mathcal{O}_K)/N(\mathfrak{a}) = N(\mathfrak{a})^{n-1}$. Note that the norm of the ideal \mathfrak{p}^m is a power of the prime integer $p \in \mathbf{Z}$ such that $\mathfrak{p} \mid p$. \square

THEOREM 4.6. *For the ideal $Tr_{K/k}(\mathcal{O}_K)$ of \mathcal{O}_k , we have*

$$H^1(G, \mathcal{O}_K)_c \cong H_c(Tr_{K/k}(\mathcal{O}_K)).$$

Proof. We borrow the idea used in the proof of Theorem 2 in [10]. We will show that $H_c(\mathfrak{b}) = 0$ for any ideal \mathfrak{b} in \mathcal{O}_k which is relatively prime to $Tr_{K/k}(\mathcal{O}_K)$. Let us denote by the ideal $Tr_{K/k}(\mathcal{O}_K)$ as \mathfrak{a} . Let \mathfrak{p} be a prime ideal of \mathcal{O}_k with $(\mathfrak{a}, \mathfrak{p}) = 1$. We may choose an ideal \mathfrak{b} in \mathcal{O}_k such that $(\mathfrak{a}\mathfrak{p}^m, \mathfrak{b}) = 1$ and $\mathfrak{a}\mathfrak{p}^m\mathfrak{b}$ is a principal ideal (a) in \mathcal{O}_k . Also there exists an ideal \mathfrak{c} in \mathcal{O}_k such that $(\mathfrak{a}\mathfrak{p}^m\mathfrak{b}, \mathfrak{c}) = 1$ and $\mathfrak{a}\mathfrak{c}$ is a principal ideal (a') . By Proposition 4.3, we have

$$H_c(a) \cong H_c(a') \cong H^1(G, \mathcal{O}_K)_c.$$

Then by Proposition 4.4, we have

$$H_c(\mathfrak{a}) \oplus H_c(\mathfrak{p}^m) \oplus H_c(\mathfrak{b}) \cong H_c(\mathfrak{a}) \oplus H_c(\mathfrak{c}) \cong H^1(G, \mathcal{O}_K)_c.$$

We have $|H_c(\mathfrak{p}^m)| = p^i$ divides $|H_c(\mathfrak{c})|$ for some i . But since $(a, \mathfrak{c}) = 1$, whence $p \mid a$, we have $|H_c(\mathfrak{p}^m)|$ and $|H_c(\mathfrak{c})|$ are mutually prime. Therefore we obtain $H_c(\mathfrak{p}^m) = 0$. This happens for arbitrary prime ideal \mathfrak{p} of \mathcal{O}_k prime to \mathfrak{a} and for any positive integer m , hence we also get $H_c(\mathfrak{b}) = 0$ and finally we have $H^1(G, \mathcal{O}_K)_c \cong H_c(\mathfrak{a})$. \square

REMARK 4.7. Note that $H_c(\mathcal{O}_k) = 0$ since $Z_c(\mathcal{O}_k) = B_c(\mathcal{O}_k) = \mathcal{O}_K$. If K/k is tamely ramified, we have $Tr_{K/k}(\mathcal{O}_K) = \mathcal{O}_k$ and we can also check $H^1(G, \mathcal{O}_K)_c = 0$ by Theorem 4.6.

COROLLARY 4.8. *We have*

$$H^1(G, \mathcal{O}_K)_c \cong \prod_{\mathfrak{p} | Tr_{K/k}(\mathcal{O}_K)} H_c \left(\mathfrak{p}^{\lfloor \frac{t_{\mathfrak{p}}}{e_{\mathfrak{p}}} \rfloor} \right)$$

where $e_{\mathfrak{p}}$ is the ramification index of \mathfrak{p} , and $\mathfrak{D}_{K/k} = \prod_{\mathfrak{p}} \left(\prod_{\mathfrak{P} | \mathfrak{p}} \mathfrak{P} \right)^{t_{\mathfrak{p}}}$ is the different of K over k .

Proof. Let $Tr_{K/k}(\mathcal{O}_K) = \prod_{\mathfrak{p} | Tr_{K/k}(\mathcal{O}_K)} \mathfrak{p}^{r_{\mathfrak{p}}}$. As in [7], we have $r_{\mathfrak{p}} = \lfloor t_{\mathfrak{p}}/e_{\mathfrak{p}} \rfloor$ since

$$\begin{aligned} \mathfrak{p}^r | Tr_{K/k}(\mathcal{O}_K) &\Leftrightarrow \mathfrak{p}^r | \mathfrak{D}_{K/k} \\ &\Leftrightarrow \prod_{\mathfrak{P} | \mathfrak{p}} \mathfrak{P}^{e_{\mathfrak{P}} r} | \left(\prod_{\mathfrak{P} | \mathfrak{p}} \mathfrak{P} \right)^{t_{\mathfrak{p}}} \\ &\Leftrightarrow e_{\mathfrak{p}} r \leq t_{\mathfrak{p}} \\ &\Leftrightarrow r \leq \lfloor t_{\mathfrak{p}}/e_{\mathfrak{p}} \rfloor. \end{aligned}$$

□

5. Local and global

Let \mathfrak{p} be a prime ideal in \mathcal{O}_k and \mathfrak{P} be the prime ideal of \mathcal{O}_K above \mathfrak{p} . We have completions $K_{\mathfrak{P}}$ and $k_{\mathfrak{p}}$ and the extension $K_{\mathfrak{P}}/k_{\mathfrak{p}}$ is Galois with the Galois group $G_{\mathfrak{P}}$ which is identified as the decomposition group at \mathfrak{P} in G . The ring of integers $\mathcal{O}_K, \mathcal{O}_k$ are embedded in $\mathcal{O}_{K_{\mathfrak{P}}}, \mathcal{O}_{k_{\mathfrak{p}}}$, respectively. For 1-cocycle $c \in Z^1(G, \mathcal{O}_K^\times)$, the c -twisted $G_{\mathfrak{P}}$ -module structure of \mathcal{O}_K extends to $\mathcal{O}_{K_{\mathfrak{P}}}$ so that we can consider the twisted cohomology $H^n(G_{\mathfrak{P}}, \mathcal{O}_{K_{\mathfrak{P}}})_c$ of $G_{\mathfrak{P}}$ into $\mathcal{O}_{K_{\mathfrak{P}}}$. On the other hand, as c induces a 1-cocycle $c_{\mathfrak{P}} \in Z^1(G_{\mathfrak{P}}, \mathcal{O}_{K_{\mathfrak{P}}}^\times)$ naturally through the embeddings

$$G_{\mathfrak{P}} \hookrightarrow G \xrightarrow{c} \mathcal{O}_K^\times \hookrightarrow \mathcal{O}_{K_{\mathfrak{P}}}^\times$$

so that we can regard $\mathcal{O}_{K_{\mathfrak{P}}}$ as $c_{\mathfrak{P}}$ -twisted $G_{\mathfrak{P}}$ -module and we can define the twisted cohomology $H^n(G_{\mathfrak{P}}, \mathcal{O}_{K_{\mathfrak{P}}})_{c_{\mathfrak{P}}}$ of $G_{\mathfrak{P}}$ into $\mathcal{O}_{K_{\mathfrak{P}}}$ twisted by $c_{\mathfrak{P}}$. Note that $c_{\mathfrak{P}}$ -twisted action in $\mathcal{O}_{K_{\mathfrak{P}}}$ is exactly same as c -twisted action so that we have $H^n(G_{\mathfrak{P}}, \mathcal{O}_{K_{\mathfrak{P}}})_c \cong H^n(G_{\mathfrak{P}}, \mathcal{O}_{K_{\mathfrak{P}}})_{c_{\mathfrak{P}}}$.

THEOREM 5.1. *For each dimension n we have*

$$H^n(G, \mathcal{O}_K)_c \cong \prod_{\mathfrak{p}} H^n(G_{\mathfrak{p}}, \mathcal{O}_{K_{\mathfrak{p}}})_{c_{\mathfrak{p}}}$$

where \mathfrak{p} runs through the prime ideals in \mathcal{O}_k and \mathfrak{P} is one prime ideal of \mathcal{O}_K dividing \mathfrak{p} .

REMARK 5.2. By the same assertion as Theorem 3.1, we have $H^n(G_{\mathfrak{p}}, \mathcal{O}_{K_{\mathfrak{p}}})_{c_{\mathfrak{p}}} = 0$ if $K_{\mathfrak{p}}/k_{\mathfrak{p}}$ is tamely ramified. Denote by W the set of all prime ideals \mathfrak{p} of \mathcal{O}_k which is wildly ramified in \mathcal{O}_K . Furthermore, since $Tr_{K/k}(\mathcal{O}_K) = \prod_{\mathfrak{p}} \mathfrak{p}^{\lfloor t_{\mathfrak{p}}/e_{\mathfrak{p}} \rfloor}$ by (4.3), we have

$$\mathfrak{p} \mid Tr_{K/k}(\mathcal{O}_K) \Leftrightarrow t_{\mathfrak{p}} > e_{\mathfrak{p}} - 1 \Leftrightarrow \mathfrak{p} \in W.$$

Hence the theorem states equivalently that

$$H^n(G, \mathcal{O}_K)_c \cong \prod_{\mathfrak{p} \in W} H^n(G_{\mathfrak{p}}, \mathcal{O}_{K_{\mathfrak{p}}})_{c_{\mathfrak{p}}} \cong \prod_{\mathfrak{p} \mid Tr_{K/k}(\mathcal{O}_K)} H^n(G_{\mathfrak{p}}, \mathcal{O}_{K_{\mathfrak{p}}})_{c_{\mathfrak{p}}}.$$

Proof. This proof is parallel to that of Theorem 1 in [3]. Let $\mathcal{O} = \prod_{\mathfrak{p}} \mathcal{O}_{K_{\mathfrak{p}}}$, and for each prime ideal \mathfrak{p} in \mathcal{O}_k let $\mathcal{O}(\mathfrak{p}) = \prod_{\mathfrak{P} \mid \mathfrak{p}} \mathcal{O}_{K_{\mathfrak{P}}}$. Then $\mathcal{O} = \prod_{\mathfrak{p}} \mathcal{O}(\mathfrak{p})$, and we have that \mathcal{O} and $\mathcal{O}(\mathfrak{p})$ are c -twisted G -modules. \mathcal{O}_K is diagonally imbedded in \mathcal{O} and $\mathcal{O}(\mathfrak{p})$. Since c_s is unit in \mathcal{O}_K , we have

$$\prod_{s \in G/G_{\mathfrak{p}}} c_s {}^s \mathcal{O}_{K_{\mathfrak{p}}} = \prod_{s \in G/G_{\mathfrak{p}}} {}^s \mathcal{O}_{K_{\mathfrak{p}}} = \mathcal{O}(\mathfrak{p})$$

where in the product, s varies over a system of representatives of the cosets in $G/G_{\mathfrak{p}}$. Therefore by Shapiro's lemma (of form of 34.1 in [1], p. 120), we have

$$H^n(G, \mathcal{O}(\mathfrak{p}))_c \cong H^n(G_{\mathfrak{p}}, \mathcal{O}_{K_{\mathfrak{p}}})_c.$$

Now we claim that

$$(5.1) \quad H^n(G, \mathcal{O}_K)_c \cong H^n(G, \mathcal{O})_c,$$

which leads the theorem since

$$H^n(G, \mathcal{O}_K)_c \cong \prod_{\mathfrak{p}} H^n(G, \mathcal{O}(\mathfrak{p}))_c \cong \prod_{\mathfrak{p}} H^n(G_{\mathfrak{p}}, \mathcal{O}_{K_{\mathfrak{p}}})_{c_{\mathfrak{p}}}.$$

Let $\beta \in \mathcal{O}_K$ be chosen as a generator of normal basis of K/k . Define $M = \sum_{s \in G} M_c c_s {}^s \beta$ then we have that M is twisted G -regular and $[\mathcal{O}_K : M]$ is finite, say l , as in the proof of Theorem 3.2. Also we define

$$\mathfrak{M} = \sum_{s \in G} \mathfrak{M}_c c_s {}^s \beta$$

where

$$\mathfrak{M}_c := \{\alpha \in \mathcal{O} ; c_s {}^s\alpha = \alpha \text{ for all } s \in G\}.$$

Then \mathfrak{M} and \mathfrak{M}_c are twisted G -modules, and we have $\mathcal{O}_K \cap \mathfrak{M} = M$ trivially.

On the other hand, we have $\mathcal{O} = \mathfrak{M} + \mathcal{O}_K$. Indeed, since $l\mathcal{O}_K \subset M$ and $M\mathcal{O} \subset \mathfrak{M}$, we have $l\mathcal{O}_K\mathcal{O} \subset \mathfrak{M}$. Denote by $\mathfrak{A} = l\mathcal{O}_K$ the ideal in \mathcal{O}_K then we have $\mathfrak{A}\mathcal{O} + \mathcal{O}_K \subset \mathfrak{M} + \mathcal{O}_K \subset \mathcal{O}$. It is enough to show that $\mathcal{O} \subset \mathfrak{A}\mathcal{O} + \mathcal{O}_K$. Now let $\alpha = (a_{\mathfrak{P}})_{\mathfrak{P}} \in \mathcal{O}$. By the strong approximation theorem, there exists an element b in K such that $\nu_{\mathfrak{P}}(b - a_{\mathfrak{P}}) \geq \nu_{\mathfrak{P}}(\mathfrak{A})$ for prime ideals $\mathfrak{P} \mid \mathfrak{A}$ and $\nu_{\mathfrak{P}}(b) \geq 0$ for prime ideals \mathfrak{P} not dividing \mathfrak{A} . Then $b \in \mathcal{O}_K$. Let $\gamma = b - \alpha \in \mathcal{O}$ then $\gamma \in \prod_{\mathfrak{P}} \mathfrak{P}^{\nu_{\mathfrak{P}}(\mathfrak{A})} \subset \mathfrak{A}\mathcal{O}$. So we have $\alpha = \gamma + b \in \mathfrak{A}\mathcal{O} + \mathcal{O}_K$ and hence $\mathcal{O} = \mathfrak{M} + \mathcal{O}_K$.

Therefore we have twisted G -module isomorphisms

$$(5.2) \quad \frac{\mathcal{O}}{\mathfrak{M}} \cong \frac{\mathfrak{M} + \mathcal{O}_K}{\mathfrak{M}} \cong \frac{\mathcal{O}_K}{\mathcal{O}_K \cap \mathfrak{M}} \cong \frac{\mathcal{O}_K}{M}.$$

Since M and \mathfrak{M} are twisted G -regular, we have $H^n(G, M)_c = 0$ and $H^n(G, \mathfrak{M})_c = 0$ so that $H^n(G, \mathcal{O}_K)_c \cong H^n(G, \mathcal{O}_K/M)_c$ and $H^n(G, \mathcal{O})_c \cong H^n(G, \mathcal{O}/\mathfrak{M})_c$ from the exact sequences $0 \rightarrow M \rightarrow \mathcal{O}_K \rightarrow \mathcal{O}_K/M \rightarrow 0$ and $0 \rightarrow \mathfrak{M} \rightarrow \mathcal{O} \rightarrow \mathcal{O}/\mathfrak{M} \rightarrow 0$. Combined with (5.2), we have $H^n(G, \mathcal{O}_K)_c \cong H^n(G, \mathcal{O})_c$. \square

Now we consider the twisted 1-cohomology for the completion $K_{\mathfrak{P}}$ and $k_{\mathfrak{p}}$ at a prime \mathfrak{P} which is wildly ramified over the extension K/k . Fix prime elements π and Π in $\mathcal{O}_{k_{\mathfrak{p}}}$ and $\mathcal{O}_{K_{\mathfrak{P}}}$, respectively, so that $\pi = \Pi^{e_{\mathfrak{p}}}$ and $\mathfrak{p} = (\pi)$ and $\mathfrak{P} = (\Pi)$ where $e_{\mathfrak{p}}$ is the ramification index of \mathfrak{p} . As in [6], the 1-cohomology of the unit group $H^1(G_{\mathfrak{P}}, \mathcal{O}_{K_{\mathfrak{P}}}^{\times})$ is the cyclic group of order $e_{\mathfrak{p}}$, generated by the canonical cohomology class $\gamma_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}$ with a representative cocycle c_{0_s} such that ${}^s\Pi = \Pi c_{0_s}$. Let $c_{\mathfrak{P}}$ be the 1-cocycle of $\mathcal{O}_{K_{\mathfrak{P}}}^{\times}$ induced by $c \in Z^1(G, \mathcal{O}_K)$ and $\gamma = [c_{\mathfrak{P}}] \in H^1(G_{\mathfrak{P}}, \mathcal{O}_{K_{\mathfrak{P}}})$. We have $\gamma = (\gamma_{K_{\mathfrak{P}}/k_{\mathfrak{p}}})^m$ for some m such that $0 \leq m < e_{\mathfrak{p}}$. We may regard $c_{\mathfrak{P}}$ as c_0^m by Proposition 2.1.

For an ideal \mathfrak{a} of $\mathcal{O}_{k_{\mathfrak{p}}}$, let

$$\begin{aligned} Z_{c_{\mathfrak{P}}}(\mathfrak{a}) &= \{\alpha \in \mathcal{O}_{K_{\mathfrak{P}}} ; \alpha \equiv c_{\mathfrak{P}_s} {}^s\alpha \pmod{\mathfrak{a}\mathcal{O}_{K_{\mathfrak{P}}}}\} \\ B_{c_{\mathfrak{P}}}(\mathfrak{a}) &= \{\alpha \in \mathcal{O}_{K_{\mathfrak{P}}} ; \alpha \equiv b \pmod{\mathfrak{a}\mathcal{O}_{K_{\mathfrak{P}}}} \text{ for some } b \in M_{c_{\mathfrak{P}}}\} = M_{c_{\mathfrak{P}}} + \mathfrak{a}\mathcal{O}_{K_{\mathfrak{P}}} \end{aligned}$$

and $H_{c_{\mathfrak{P}}}(\mathfrak{a}) = Z_{c_{\mathfrak{P}}}(\mathfrak{a})/B_{c_{\mathfrak{P}}}(\mathfrak{a})$ where $M_{c_{\mathfrak{P}}} = Z^0(G_{\mathfrak{P}}, \mathcal{O}_{K_{\mathfrak{P}}})_{c_{\mathfrak{P}}}$. Then we have very similar arguments with propositions in section 4 for the local case.

Since the set of trace image $Tr_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}(\mathcal{O}_{K_{\mathfrak{P}}})$ is an ideal in $\mathcal{O}_{k_{\mathfrak{p}}}$, there is a nonnegative integer $r_{\mathfrak{p}}$ such that $Tr_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}(\mathcal{O}_{K_{\mathfrak{P}}}) = \mathfrak{p}^{r_{\mathfrak{p}}} = (\pi^{r_{\mathfrak{p}}})$. It is known that $r_{\mathfrak{p}} = \lfloor t_{\mathfrak{p}}/e_{\mathfrak{p}} \rfloor$ where $\mathfrak{P}^{t_{\mathfrak{p}}} = \mathfrak{D}_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}$, and $\mathfrak{D}_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}$ is the different of $K_{\mathfrak{P}}$ over $k_{\mathfrak{p}}$ because of the same argument with (4.3). We have $H^1(G_{\mathfrak{P}}, \mathcal{O}_{K_{\mathfrak{P}}})_{c_{\mathfrak{P}}} \cong H_{c_{\mathfrak{P}}}(\pi^{r_{\mathfrak{p}}}) = H_{c_{\mathfrak{P}}}(Tr_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}(\mathcal{O}_{K_{\mathfrak{P}}}))$ because propositions 4.1, 4.2, and 4.3 hold naturally in the local case.

Since we have the tower

$$\pi^{r_{\mathfrak{p}}}\mathcal{O}_{K_{\mathfrak{P}}} = \mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}} \subset B_{c_{\mathfrak{P}}}(\pi^{r_{\mathfrak{p}}}) \subset Z_{c_{\mathfrak{P}}}(\pi^{r_{\mathfrak{p}}}) \subset \mathcal{O}_{K_{\mathfrak{P}}},$$

we have

$$[Z_{c_{\mathfrak{P}}}(\pi^{r_{\mathfrak{p}}}) : B_{c_{\mathfrak{P}}}(\pi^{r_{\mathfrak{p}}})] \mid N(\mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}})/[B_{c_{\mathfrak{P}}}(\pi^{r_{\mathfrak{p}}}) : \mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}}] .$$

As in [6], we have $\Pi^m M_{c_{\mathfrak{P}}} = \mathfrak{p}$ if $m \neq 0$ and $M_{c_{\mathfrak{P}}} = \mathcal{O}_{k_{\mathfrak{p}}}$ if $m = 0$. Therefore we have $\Pi^m B_{c_{\mathfrak{P}}}(\pi^{r_{\mathfrak{p}}}) = \Pi^m M_{c_{\mathfrak{P}}} + \Pi^m \mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}} = \mathfrak{p} + \mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}+m}$ if $m \neq 0$ and $B_{c_{\mathfrak{P}}}(\pi^{r_{\mathfrak{p}}}) = \mathcal{O}_{k_{\mathfrak{p}}} + \mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}}$ if $m = 0$. Now

$$[B_{c_{\mathfrak{P}}}(\pi^{r_{\mathfrak{p}}}) : \mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}}] = \left| \frac{\mathfrak{p} + \mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}+m}}{\mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}+m}} \right| = \left| \frac{\mathfrak{p}}{\mathfrak{p} \cap \mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}+m}} \right| = \left| \frac{\mathfrak{p}}{\mathfrak{p}^{r_{\mathfrak{p}}+1}} \right| = N(\mathfrak{p})^{r_{\mathfrak{p}}}$$

for $m \neq 0$, and

$$[B_{c_{\mathfrak{P}}}(\pi^{r_{\mathfrak{p}}}) : \mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}}] = \left| \frac{\mathcal{O}_{k_{\mathfrak{p}}} + \mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}}}{\mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}}} \right| = \left| \frac{\mathcal{O}_{k_{\mathfrak{p}}}}{\mathcal{O}_{k_{\mathfrak{p}}} \cap \mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}}} \right| = \left| \frac{\mathcal{O}_{k_{\mathfrak{p}}}}{\mathfrak{p}^{r_{\mathfrak{p}}}} \right| = N(\mathfrak{p})^{r_{\mathfrak{p}}}$$

for $m = 0$. Since $N(\mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}}) = N(\mathfrak{p})^{e_{\mathfrak{p}}f_{\mathfrak{p}}r_{\mathfrak{p}}}$, finally we have $N(\mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}})/[B_{c_{\mathfrak{P}}}(\pi^{r_{\mathfrak{p}}}) : \mathfrak{P}^{e_{\mathfrak{p}}r_{\mathfrak{p}}}] = N(\mathfrak{p})^{e_{\mathfrak{p}}f_{\mathfrak{p}}r_{\mathfrak{p}}-r_{\mathfrak{p}}}$. We state the summary in the following proposition.

PROPOSITION 5.3. *We have*

$$H^1(G_{\mathfrak{P}}, \mathcal{O}_{K_{\mathfrak{P}}})_{c_{\mathfrak{P}}} \cong H_{c_{\mathfrak{P}}}(Tr_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}(\mathcal{O}_{K_{\mathfrak{P}}})) = H_{c_{\mathfrak{P}}}(\mathfrak{p}^{\lfloor t_{\mathfrak{p}}/e_{\mathfrak{p}} \rfloor})$$

and $[B_{c_{\mathfrak{P}}}(Tr_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}(\mathcal{O}_{K_{\mathfrak{P}}})) : Tr_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}(\mathcal{O}_{K_{\mathfrak{P}}})\mathcal{O}_{K_{\mathfrak{P}}}] = N(\mathfrak{p})^{\lfloor t_{\mathfrak{p}}/e_{\mathfrak{p}} \rfloor}$. The order $|H^1(G_{\mathfrak{P}}, \mathcal{O}_{K_{\mathfrak{P}}})_{c_{\mathfrak{P}}}|$ is a divisor of $N(\mathfrak{p})^{(e_{\mathfrak{p}}f_{\mathfrak{p}}-1)\lfloor t_{\mathfrak{p}}/e_{\mathfrak{p}} \rfloor}$.

6. Quadratic fields

Let $K = \mathbf{Q}(\sqrt{m})$ be a quadratic number field where m is square-free, and let $k = \mathbf{Q}$. Since the Galois group $G = \langle s \rangle$ is cyclic, we have $|H^n(G, \mathcal{O}_K)_c| = |M_c/P_c|$ for all integer n and for all cocycle $c \in Z^1(G, \mathcal{O}_K^\times)$. Hence for all dimension n , we have $|H^n(G, \mathcal{O}_K)_c| = 1$ if and only if (a) m is congruent to 1 modulo 4, or (b) $m \equiv 3 \pmod{4}$, $m > 0$, $c \sim \pm\varepsilon$, and v is odd, where $\varepsilon = u + v\sqrt{m}$ is the fundamental unit of norm 1 in \mathcal{O}_K , or equivalently (b') $m \equiv 3 \pmod{4}$, $m > 0$, $c \sim \pm\varepsilon$,

and the central entry $a_{r/2}$ of the simple continued fraction expansion of $\sqrt{m} = [a_0; \overline{a_1, \dots, a_r}]$ is odd (see [4]). In other words, for $m \equiv 3 \pmod{4}$, $|H^n(G, \mathcal{O}_K)_c| = 1$ if m_2 is odd and $|H^n(G, \mathcal{O}_K)_c| = 2$ if m_2 is even, where $c \sim \frac{s \prod m_2}{\prod m_2}$ as 1-cocycle in $Z^1(G_{\mathfrak{P}}, \mathcal{O}_{K_{\mathfrak{P}}})$ in the completion at $\mathfrak{P} \mid 2$ (see [7]).

This is verified by computing the 1-twisted cohomology in two explicit ways. First, let $\Delta_c = 1 - c_s s$ and $N_c = 1 + c_s s$, and denote by $M^c = \ker N_c$ and $P^c = \text{im } N$ then we have $M^c = M_{-c}$ and $P^c = P_{-c}$ clearly. On the other hand, as the use of theorem 4.6, we compute $H_c(\text{Tr}_{K/\mathbf{Q}}(\mathcal{O}_K))$: We may assume $m \equiv 2, 3 \pmod{4}$ for considering the wild ramified case only. We have $B_c(\text{Tr}_{K/\mathbf{Q}}(\mathcal{O}_K)) = M_c + \text{Tr}_{K/\mathbf{Q}}(\mathcal{O}_K)\mathcal{O}_K = M_c + 2\mathcal{O}_K$, which satisfies

$$[B_c(\text{Tr}_{K/\mathbf{Q}}(\mathcal{O}_K)) : \text{Tr}_{K/\mathbf{Q}}(\mathcal{O}_K)\mathcal{O}_K] = 2$$

by Proposition 4.5. To determine $Z_c(\text{Tr}_{K/\mathbf{Q}}(\mathcal{O}_K))$ when $c \sim \pm\varepsilon$, let $x + y\sqrt{m}$ be satisfying $\pm\varepsilon(x - y\sqrt{m}) - (x + y\sqrt{m}) \in 2\mathcal{O}_K$, equivalently,

$$(6.1) \quad 2 \mid x(u \pm 1) + yvm, \quad 2 \mid xv - y(u \pm 1).$$

If v is even, the condition (6.1) gives nothing, so we have $Z_c(\text{Tr}_{K/\mathbf{Q}}(\mathcal{O}_K)) = \mathcal{O}_K$ and $|H^1(G, \mathcal{O}_K)_c| = 2$. In the case v is odd, which happens when $m \equiv 3 \pmod{4}$ only, we obtain $x \equiv y \pmod{2}$ so we have $[Z_c(\text{Tr}_{K/\mathbf{Q}}(\mathcal{O}_K)) : \text{Tr}_{K/\mathbf{Q}}(\mathcal{O}_K)\mathcal{O}_K] = 2$ and hence $|H^1(G, \mathcal{O}_K)_c| = 1$. For $c \sim \pm 1$, it is trivial to see $Z_c(\text{Tr}_{K/\mathbf{Q}}(\mathcal{O}_K)) = \mathcal{O}_K$ and $|H^1(G, \mathcal{O}_K)_c| = 2$.

7. Biquadratic fields

Here we give an example of the twisted cohomology of a non-cyclic extension. Let $K = \mathbf{Q}(\sqrt{m_1}, \sqrt{m_2})$ be a biquadratic field where m_1, m_2 are distinct and squarefree. K has three subfields $k_1 = \mathbf{Q}(\sqrt{m_1})$, $k_2 = \mathbf{Q}(\sqrt{m_2})$, and $k_3 = \mathbf{Q}(\sqrt{m_3})$ where $m_3 = \frac{m_1 m_2}{d}$ and $d = \text{gcd}(m_1, m_2)$. The Galois group of K over \mathbf{Q} is $G = \text{Gal}(K/\mathbf{Q}) = \{1, s_1, s_2, s_3\}$ where $s_i s_j = s_k$ with i, j, k distinct, $s_i^2 = 1$ for all i , and for each i s_i acts trivially on the subfield k_i , not on k_j for $j \neq i$. It is known that any of biquadratic fields falls into the following three types:

[Type I] $m_1 \equiv m_2 \equiv m_3 \equiv 1 \pmod{4}$ with

$$\mathcal{O}_K = \left[1, \frac{1 + \sqrt{m_1}}{2}, \frac{1 + \sqrt{m_2}}{2}, \left(\frac{1 + \sqrt{m_1}}{2} \right) \left(\frac{1 + \sqrt{m_2}}{2} \right) \right]_{\mathbf{Z}}.$$

[**Type II**] $m_1 \equiv 1$ and $m_2 \equiv m_3 \equiv 2, 3 \pmod{4}$ with

$$\mathcal{O}_K = \left[1, \frac{1 + \sqrt{m_1}}{2}, \sqrt{m_2}, \frac{\sqrt{m_2} + \sqrt{m_3}}{2} \right]_{\mathbf{Z}}.$$

[**Type III**] $m_1 \equiv 3$ and $m_2 \equiv m_3 \equiv 2 \pmod{4}$ with

$$\mathcal{O}_K = \left[1, \sqrt{m_1}, \sqrt{m_2}, \frac{\sqrt{m_2} + \sqrt{m_3}}{2} \right]_{\mathbf{Z}}.$$

The unit group \mathcal{O}_K^\times has one or three fundamental units if only one or three of m_1, m_2, m_3 are positive, respectively. Denote by $\varepsilon_1, \varepsilon_2, \varepsilon_3$ the fundamental units of k_1, k_2, k_3 respectively, with norm 1, if it is the case that ones exist. We also denote $\varepsilon_i = u_i + v_i\omega_i$ for $u_i, v_i \in \mathbf{Z}$ and $\omega_i = \frac{1 + \sqrt{m_i}}{2}$ if $m_i \equiv 1 \pmod{4}$, $\omega_i = \sqrt{m_i}$ if $m_i \equiv 2, 3 \pmod{4}$.

For a 1-cocycle $c \in Z^1(G, \mathcal{O}_K^\times)$, M_c/P_c for the biquadratic fields is computed along the 2-adic property of $\gamma = [c]$ in [2] and we give an explicit computation of M_c/P_c and $H^1(G, \mathcal{O}_K)_c$ globally for certain cocycles c in \mathcal{O}_K^\times , which inherit cocycles in the unit group of quadratic subfields, and we compare with the case of the quadratic field. We may consider only the cases [Type II] and [Type III] because in [Type I], we have $\text{Tr}_{K/\mathbf{Q}}(\mathcal{O}_K) = \mathbf{Z}$ so that the extension is tamely ramified, hence all the twisted cohomologies vanish.

For $i = 1, 2, 3$, define $c_i : G \rightarrow \mathcal{O}_K^\times$ as $c_{i1} = 1$, $c_{is_i} = 1$ and $c_{is_j} = \varepsilon_i$, $j \neq i$ then c_i is a 1-cocycle into \mathcal{O}_K^\times . Since the nontrivial action on k_i by $G_i = \text{Gal}(k_i/\mathbf{Q})$ is the same as ones by s_j for $j \neq i$, we may regard in the quadratic field k_i $s_j = s_k \in G_i$ and $c_i \in Z^1(G_i, \mathcal{O}_{k_i}^\times)$. Denote by

$$\begin{aligned} M_{c_i}(\mathcal{O}_K) &= M_{c_i} = Z^0(G, \mathcal{O}_K)_{c_i}, \\ M_{c_i}(\mathcal{O}_{k_i}) &= Z^0(G_i, \mathcal{O}_{k_i})_{c_i}, \\ P_{c_i}(\mathcal{O}_K) &= P_{c_i} = \{p_{c_i \mathcal{O}_K}(\alpha) = \sum_{s \in G} c_s {}^s\alpha; \alpha \in \mathcal{O}_K\}, \text{ and} \\ P_{c_i}(\mathcal{O}_{k_i}) &= \{p_{c_i \mathcal{O}_{k_i}}(\alpha) = \alpha + \varepsilon_i {}^{s_j}\alpha; \alpha \in \mathcal{O}_{k_i}\}. \end{aligned}$$

7.1. 0-dimensional twisted cohomology

Let $\alpha \in M_{c_i}(\mathcal{O}_K)$. Since ${}^{s_i}\alpha = \alpha$ if and only if $\alpha \in \mathcal{O}_{k_i}$, we have

$$(7.1) \quad M_{c_i}(\mathcal{O}_K) = M_{c_i}(\mathcal{O}_{k_i}).$$

We will compute $P_{c_i}(\mathcal{O}_K)$: First note that $\alpha + {}^s i \alpha \in \mathcal{O}_{k_i}$ and

$$\begin{aligned} p_{c_i \mathcal{O}_K}(\alpha) &= \sum_{s \in G} c_s {}^s \alpha = \alpha + {}^s i \alpha + \varepsilon_i {}^s j \alpha + \varepsilon_i {}^s k \alpha \\ &= (\alpha + {}^s i \alpha) + \varepsilon_i {}^s j (\alpha + {}^s i \alpha) \\ &= p_{c_i \mathcal{O}_{k_i}}(\alpha + {}^s i \alpha). \end{aligned}$$

[Type II] Let $\alpha = x + y \left(\frac{1 + \sqrt{m_1}}{2} \right) + z\sqrt{m_2} + w \left(\frac{\sqrt{m_2} + \sqrt{m_3}}{2} \right)$ with $x, y, z, w \in \mathbf{Z}$. Then we have

$$\alpha + {}^s i \alpha = \begin{cases} 2x + 2y \left(\frac{1 + \sqrt{m_1}}{2} \right), & i = 1 \\ (2x + y) + (2z + w)\sqrt{m_2}, & i = 2 \\ (2x + y) + w\sqrt{m_3}, & i = 3. \end{cases}$$

So we have

$$P_{c_1}(\mathcal{O}_K) = 2P_{c_1}(\mathcal{O}_{k_1}), \quad P_{c_2}(\mathcal{O}_K) = P_{c_2}(\mathcal{O}_{k_2}), \quad P_{c_3}(\mathcal{O}_K) = P_{c_3}(\mathcal{O}_{k_3}).$$

Hence if $m_2 \equiv m_3 \equiv 3 \pmod{4}$,

$$\begin{aligned} \left| \frac{M_{c_1}}{P_{c_1}} \right| &= 2 \\ \left| \frac{M_{c_2}}{P_{c_2}} \right| &= \begin{cases} 1 & \text{if } v_2 \text{ is odd} \\ 2 & \text{if } v_2 \text{ is even} \end{cases} \\ \left| \frac{M_{c_3}}{P_{c_3}} \right| &= \begin{cases} 1 & \text{if } v_3 \text{ is odd} \\ 2 & \text{if } v_3 \text{ is even} \end{cases} \end{aligned}$$

while if $m_2 \equiv m_3 \equiv 2 \pmod{4}$, it is shown in [2] that $\left| \frac{M_c}{P_c} \right| = 2$ for all cocycles c .

[Type III] Let $\alpha = x + y\sqrt{m_1} + z\sqrt{m_2} + w \left(\frac{\sqrt{m_2} + \sqrt{m_3}}{2} \right)$ with $x, y, z, w \in \mathbf{Z}$.

Then

$$\alpha + {}^s i \alpha = \begin{cases} 2x + 2y\sqrt{m_1}, & i = 1 \\ 2x + (2z + w)\sqrt{m_2}, & i = 2 \\ 2x + w\sqrt{m_3}, & i = 3. \end{cases}$$

We obtain that

$$P_{c_1}(\mathcal{O}_K) = 2P_{c_1}(\mathcal{O}_{k_1}), \quad \text{and } P_{c_i}(\mathcal{O}_K) = p_{c_i \mathcal{O}_{k_i}}([2, \sqrt{m_i}] \mathbf{Z}) \subset P_{c_i}(\mathcal{O}_{k_i}),$$

for $i = 2, 3$. For $i = 2, 3$, let $\xi_i \in \mathcal{O}_{k_i}$ be an integer in k_i such that $c_{i s_j} = \frac{{}^s j \xi_i}{\xi_i}$. We may choose $\xi_i = 1 + {}^s j \varepsilon_i = (u_i + 1) - v_i \sqrt{m_i}$. Then

$\xi_i P_{c_i}(\mathcal{O}_{k_i}) = \text{Tr}_{k_i/\mathbf{Q}}(\xi_i \mathcal{O}_{k_i})$ and
 $\xi_i P_{c_i \mathcal{O}_{k_i}}([2, \sqrt{m_i}]_{\mathbf{Z}}) = \text{Tr}_{k_i/\mathbf{Q}}(\xi_i [2, \sqrt{m_i}]_{\mathbf{Z}})$. It is easy to see that

$$(7.2) \quad \text{Tr}_{k_i/\mathbf{Q}}(\xi_i \mathcal{O}_{k_i}) = \{2((u_i + 1)x - yv_i m_i) ; x, y \in \mathbf{Z}\}$$

$$(7.3) \quad = 2 \gcd(u_i + 1, v_i m_i) \mathbf{Z}$$

$$(7.4) \quad \text{Tr}_{k_i/\mathbf{Q}}(\xi_i [2, \sqrt{m_i}]_{\mathbf{Z}}) = \{2(2(u_i + 1)x - yv_i m_i) ; x, y \in \mathbf{Z}\}$$

$$(7.5) \quad = 2 \gcd(2(u_i + 1), v_i m_i) \mathbf{Z}.$$

Hence we have

$$p_{c_i \mathcal{O}_{k_i}}([2, \sqrt{m_i}]_{\mathbf{Z}}) = \begin{cases} P_{c_i}(\mathcal{O}_{k_i}) & \text{if } D_i \text{ is odd,} \\ 2P_{c_i}(\mathcal{O}_{k_i}) & \text{if } D_i \text{ is even,} \end{cases}$$

if we denote by

$$(7.6) \quad D_i = \frac{v_i}{\gcd(u_i + 1, v_i)}.$$

As a result, we have

$$\left| \frac{M_{c_1}}{P_{c_1}} \right| = \begin{cases} 2 & \text{if } v_1 \text{ is odd} \\ 4 & \text{if } v_1 \text{ is even} \end{cases}$$

$$\left| \frac{M_{c_i}}{P_{c_i}} \right| = \begin{cases} 2 & \text{if } D_i \text{ is odd} \\ 4 & \text{if } D_i \text{ is even} \end{cases} \quad (i = 2, 3).$$

We can rewrite the condition for $i = 2, 3$ as

$$\left| \frac{M_{c_i}}{P_{c_i}} \right| = \begin{cases} 4 & \text{if } u_i \equiv 1, v_i \equiv 0 \pmod{4} \\ 2 & \text{otherwise} \end{cases} \quad (i = 2, 3).$$

Indeed, if $u_i \equiv 1, v_i \equiv 0 \pmod{4}$, $\gcd(u_i + 1, v_i) = 2$ so trivially D_i is even. Also it is easy to see that if $v_i \equiv 2 \pmod{4}$ then D_i is odd for any odd u_i . The case remained is that $u_i \equiv 3, v_i \equiv 0 \pmod{4}$. Let $\nu \geq 2$ be satisfying $2^\nu \parallel v_i$. By Lemma 7 in [4], we have $u_i + 1 \equiv 0$ or $2 \pmod{2^{\nu+1}}$, and hence we have that D_i is odd.

REMARK 7.1. For $m_i \equiv 2 \pmod{4}$, squarefree, with the norm of fundamental unit $N(\varepsilon_i) = 1$, the both parity of D_i occur, while the odd case appears more frequently. Up to 500, we have m_i 's with odd D_i :

6, 14, 22, 30, 34, 38, 42, 46, 62, 70, 78, 86, 94, 102, 110, 118, 134, 138, 142, 154, 158, 166, 174, 182, 186, 190, 194, 206, 210, 214, 222, 230, 238, 246, 254, 262, 266, 278, 282, 286, 302, 310, 318, 322, 326,

330, 334, 358, 366, 374, 382, 386, 390, 398, 406, 422, 426, 430, 434,
438, 446, 454, 462, 470, 474, 478, 482, 494,

and m_i 's with even D_i (that is, $u_i \equiv 1, v_i \equiv 0 \pmod{4}$):
66, 114, 146, 178, 258, 354, 402, 410, 418, 466, 498.

7.2. 1-dimensional twisted cohomology

We compute $Z_{c_i}(Tr_{K/\mathbf{Q}}(\mathcal{O}_K))$ in order to get the index
 $[Z_{c_i}(Tr_{K/\mathbf{Q}}(\mathcal{O}_K)) : Tr_{K/\mathbf{Q}}(\mathcal{O}_K)\mathcal{O}_K]$, while

$$|H^1(G, \mathcal{O}_K)_c| = [Z_{c_i}(Tr_{K/\mathbf{Q}}(\mathcal{O}_K)) : Tr_{K/\mathbf{Q}}(\mathcal{O}_K)\mathcal{O}_K] / |\mathcal{O}_K / Tr_{K/\mathbf{Q}}(\mathcal{O}_K)|$$

by Proposition 4.5. For the condition to $\alpha \in Z_{c_i}(Tr_{K/\mathbf{Q}}(\mathcal{O}_K))$, note that for distinct i, j , and k , when $c_{is_i} {}^{s_i}\alpha - \alpha = {}^{s_i}\alpha - \alpha \in Tr_{K/\mathbf{Q}}(\mathcal{O}_K)\mathcal{O}_K$, since $\varepsilon_i {}^{s_j}\alpha - \alpha = {}^{s_i}(\varepsilon_i {}^{s_k}\alpha - \alpha) + {}^{s_i}\alpha - \alpha$, we have

$$c_{is_j} {}^{s_j}\alpha - \alpha = \varepsilon_i {}^{s_j}\alpha - \alpha \in Tr_{K/\mathbf{Q}}(\mathcal{O}_K)\mathcal{O}_K$$

if and only if

$$c_{is_k} {}^{s_k}\alpha - \alpha = \varepsilon_i {}^{s_k}\alpha - \alpha \in Tr_{K/\mathbf{Q}}(\mathcal{O}_K)\mathcal{O}_K,$$

hence it is enough to check cocycle images of only two automorphisms, one with the same index i and one any other.

[Type II] Note that $Tr_{K/\mathbf{Q}}(\mathcal{O}_K) = 2\mathbf{Z}$ and

$[B_c(Tr_{K/\mathbf{Q}}(\mathcal{O}_K)) : Tr_{K/\mathbf{Q}}(\mathcal{O}_K)\mathcal{O}_K] = 2$ for all 1-cocycles c by Proposition 4.5. Denote by the integral basis elements $\eta_1 = 1, \eta_2 = \frac{1+\sqrt{m_1}}{2}, \eta_3 = \sqrt{m_2}, \eta_4 = \frac{\sqrt{m_2+\sqrt{m_3}}}{2}$ so that $\mathcal{O}_K = [\eta_1, \eta_2, \eta_3, \eta_4]\mathbf{Z}$.

(a) $c = c_1$.

Let $\alpha = x\eta_1 + y\eta_2 + z\eta_3 + w\eta_4 \in Z_{c_1}(2\mathcal{O}_K)$. The condition ${}^{s_1}\alpha - \alpha \in 2\mathcal{O}_K$ holds for all integer in \mathcal{O}_K . Since

$$\begin{aligned} \varepsilon_1 {}^{s_2}\alpha - \alpha &= \left[x(u_1 - 1) + y \left\{ u_1 + v_1 \left(\frac{1 - m_1}{4} \right) \right\} \right] \eta_1 \\ &+ [xv_1 - y(u_1 + 1)] \eta_2 \\ &+ \left[z \left\{ u_1 - 1 - v_1 \left(\frac{d-1}{2} \right) \right\} \right. \\ &+ \left. w \left\{ u_1 + v_1 \left(\frac{2 - m_1/d - d}{4} \right) \right\} \right] \eta_3 \\ &+ \left[zdv_1 + w \left\{ v_1 \left(\frac{d-1}{2} \right) - u_1 - 1 \right\} \right] \eta_4, \end{aligned}$$

where $\frac{1-m_1}{4}$, $\frac{d-1}{2}$, and $\frac{2-m_1/d-d}{4}$ are integers, we have the system of congruence equations

$$(7.7) \quad x(u_1 - 1) + y(u_1 + v_1(1 - m_1)/4) \equiv 0 \pmod{2}$$

$$(7.8) \quad xv_1 - y(u_1 + 1) \equiv 0 \pmod{2}$$

$$(7.9) \quad z(u_1 - 1 - v_1(d - 1)/2) + w(u_1 + v_1(2 - m_1/d - d)/4) \equiv 0 \pmod{2}$$

$$(7.10) \quad zv_1 + w(v_1(d - 1)/2 - u_1 - 1) \equiv 0 \pmod{2}$$

for d is odd as a divisor of m_1 . We separate the cases along to the parity of u_1 and v_1 :

- (i) Suppose v_1 is even. Then u_1 is odd and the system of equations turns out to be $y \equiv w \equiv 0 \pmod{2}$. We have $[Z_{c_1}(2\mathcal{O}_K) : 2\mathcal{O}_K] = 4$ and hence $|H^1(G, \mathcal{O}_K)_{c_1}| = [Z_{c_1}(2\mathcal{O}_K) : B_{c_1}(2\mathcal{O}_K)] = 2$.
- (ii) Suppose v_1 is odd and u_1 is even. Note the fact that v_1 is odd implies $m_1 \equiv 5 \pmod{8}$ since if we assume $m_1 \equiv 1 \pmod{8}$ then $\frac{m_1-1}{4}$ is even and $1 = N(\varepsilon_1) = u_1(u_1 + v_1) - v_1^2 \frac{m_1-1}{4} \equiv u_1(u_1 + 1) \equiv 0 \pmod{2}$ makes a contradiction. Now since $m_1 \equiv 5 \pmod{8}$, we have that $\frac{m_1-1}{4}$ is odd and $\frac{2-m_1/d-d}{4} \equiv \frac{d+1}{2} \pmod{2}$. Indeed, since d is odd we have $d^2 \equiv 1 \pmod{8}$ so $m_1/d \equiv m_1d \pmod{8}$ and

$$\begin{aligned} 2 - m_1/d - d &\equiv 2 - 5d - d \pmod{8} \\ &\equiv 2 - 6(d + 1) + 6 \pmod{8} \\ &\equiv 8 - 12 \left(\frac{d+1}{2} \right) \pmod{8} \\ &\equiv 4 \left(\frac{d+1}{2} \right) \pmod{8}. \end{aligned}$$

The system of equations (7.7)–(7.10) becomes

$$(7.11) \quad x + y \equiv 0 \pmod{2}$$

$$(7.12) \quad z(1 - (d - 1)/2) + w(d + 1)/2 \equiv 0 \pmod{2}$$

$$(7.13) \quad z + w((d - 1)/2 - 1) \equiv 0 \pmod{2}.$$

First suppose that $(d - 1)/2$ is odd and $(d + 1)/2$ is even. The system of equation turns to $x \equiv y \pmod{2}$ and $z \equiv 0 \pmod{2}$. On the other hand, if $(d - 1)/2$ is even and $(d + 1)/2$ is odd, then the system of equation turns to $x \equiv y \pmod{2}$ and $z \equiv w \pmod{2}$. In any case, the

index of $Z_{c_1}(2\mathcal{O}_K)$ from $2\mathcal{O}_K$ is equal to 4 and hence $|H^1(G, \mathcal{O}_K)_{c_1}| = 2$.

- (iii) Suppose v_1 and u_1 are odd. We have that $\frac{m_1-1}{4}$ is odd and $\frac{2-m_1/d-d}{4} \equiv \frac{d+1}{2}$. The system of equations (7.7)–(7.10) turns to

$$(7.14) \quad x \equiv 0 \pmod{2}$$

$$(7.15) \quad -z(d-1)/2 + w(1 + (d+1)/2) \equiv 0 \pmod{2}$$

$$(7.16) \quad z + w(d-1)/2 \equiv 0 \pmod{2}.$$

If $d \equiv 1 \pmod{4}$, since $(d-1)/2$ is even and $(d+1)/2$ is odd, we obtain $x \equiv 0 \pmod{2}$ and $z \equiv 0 \pmod{2}$. If $d \equiv 3 \pmod{4}$ then we obtain $x \equiv 0 \pmod{2}$ and $z \equiv w \pmod{2}$. Either of the cases, we have $[Z_{c_1}(2\mathcal{O}_K) : 2\mathcal{O}_K] = 4$ and therefore we have $|H^1(G, \mathcal{O}_K)_{c_1}| = [Z_{c_1}(2\mathcal{O}_K) : B_{c_1}(2\mathcal{O}_K)] = 2$.

- (b) $c = c_2$.

Let $\alpha = x\eta_1 + y\eta_2 + z\eta_3 + w\eta_4 \in Z_{c_2}(2\mathcal{O}_K)$. From ${}^{s_2}\alpha - \alpha = y\eta_1 - 2y\eta_2 + w\eta_3 - 2w\eta_4 \in 2\mathcal{O}_K$, we obtain that y, w are even.

Now we have

$$\begin{aligned} \varepsilon_2 {}^{s_1}\alpha - \alpha &= \left[x(u_2 - 1) - zv_2m_2 - wv_2 \left(\frac{m_2 - m_2/d}{2} \right) \right] \eta_1 \\ &\quad + \left[y(u_2 - 1) - wv_2 \frac{m_2}{d} \right] \eta_2 \\ &\quad + \left[xv_2 - yv_2 \left(\frac{d-1}{2} \right) - z(u_2 + 1) \right] \eta_3 \\ &\quad + [yv_2d - w(u_2 + 1)] \eta_4 \\ &\in 2\mathcal{O}_K \end{aligned}$$

which yields the system of equations

$$(7.17) \quad x(u_2 - 1) - zv_2m_2 \equiv 0 \pmod{2}$$

$$(7.18) \quad xv_2 - z(u_2 + 1) \equiv 0 \pmod{2}$$

for $y, w \equiv 0 \pmod{2}$. Since we have that u_2 and v_2 have distinct parity, there are two cases:

- (i) Suppose v_2 is odd. Then m_2 is odd and u_2 is even. the system of equations (7.17)–(7.18) turns to be $x \equiv z \pmod{2}$. So $Z_{c_2}(2\mathcal{O}_K)$ is the set of elements $x\eta_1 + y\eta_2 + z\eta_3 + w\eta_4$ with $x \equiv z, y \equiv w \equiv 0 \pmod{2}$ and has index 2 from $2\mathcal{O}_K$. Therefore we have $|H^1(G, \mathcal{O}_K)_{c_2}| = 1$.

- (ii) Suppose v_2 is even. The system of equations (7.17)–(7.18) does not give any restricting conditions. So we have $Z_{c_2}(2\mathcal{O}_K)$ is the set of elements $x\eta_1 + y\eta_2 + z\eta_3 + w\eta_4$ with $y \equiv w \equiv 0 \pmod{2}$ with index 4 from $2\mathcal{O}_K$ and hence $|H^1(G, \mathcal{O}_K)_{c_2}| = 2$.
- (c) $c = c_3$.
Let $\alpha = x\eta_1 + y\eta_2 + z\eta_3 + w\eta_4 \in Z_{c_3}(2\mathcal{O}_K)$. From ${}^{s_3}\alpha - \alpha = y\eta_1 - 2y\eta_2 - (2z + w)\eta_3 \in 2\mathcal{O}_K$ we have $y \equiv w \equiv 0 \pmod{2}$.
Now

$$\begin{aligned} \varepsilon_3 {}^{s_1}\alpha - \alpha &= \left[x(u_3 - 1) + zv_3 \left(\frac{m_2}{d} \right) + wv_3 \left(\frac{m_2/d - m_3}{2} \right) \right] \eta_1 \\ &\quad + \left[y(u_3 - 1) - 2zv_3 \left(\frac{m_2}{d} \right) - wv_3 \left(\frac{m_2}{d} \right) \right] \eta_2 \\ &\quad + \left[-xv_3 + yv_3 \left(\frac{m_1/d - 1}{2} \right) - z(u_3 + 1) \right] \eta_3 \\ &\quad + [2xv_3 + yv_3 - w(u_3 + 1)] \eta_4 \end{aligned}$$

where $\frac{m_2}{d}$, $\frac{m_2/d - m_3}{2}$, $\frac{m_1/d - 1}{2}$ are integers. Since $y \equiv w \equiv 0 \pmod{2}$, we obtain the system of congruence equations

$$(7.19) \quad x(u_3 - 1) + zv_3(m_2/d) \equiv 0 \pmod{2}$$

$$(7.20) \quad xv_3 + z(u_3 + 1) \equiv 0 \pmod{2}$$

- (i) Suppose v_3 is odd. Then m_3 and m_2 are odd, and we obtain from the system of equations (7.19)–(7.20) that $x + z \equiv 0 \pmod{2}$. So we have $Z_{c_3}(2\mathcal{O}_K)$ is the set of elements $x\eta_1 + y\eta_2 + z\eta_3 + w\eta_4$ with $x \equiv z$, $y \equiv w \equiv 0 \pmod{2}$ with index 2 from $2\mathcal{O}_K$ and hence $|H^1(G, \mathcal{O}_K)_{c_3}| = 1$.
- (ii) Suppose v_3 is even. We obtain no more conditions other than $y \equiv w \equiv 0 \pmod{2}$ from the system of equations (7.19)–(7.20). $|H^1(G, \mathcal{O}_K)_{c_3}| = 2$.

[Type III] Note that $\text{Tr}_{K/\mathbf{Q}}(\mathcal{O}_K) = 4\mathbf{Z}$ and we have $[B_c(4\mathcal{O}_K) : 4\mathcal{O}_K] = 4$ for all cocycles c by Proposition 4.5. Denote by the integral basis elements $\eta_1 = 1$, $\eta_2 = \sqrt{m_1}$, $\eta_3 = \sqrt{m_2}$, $\eta_4 = \frac{\sqrt{m_2} + \sqrt{m_3}}{2}$ so that $\mathcal{O}_K = [\eta_1, \eta_2, \eta_3, \eta_4]\mathbf{Z}$.

- (a) $c = c_1$.

Let $\alpha = x\eta_1 + y\eta_2 + z\eta_3 + w\eta_4 \in Z_{c_1}(4\mathcal{O}_K)$. From ${}^{s_1}\alpha - \alpha = -2z\eta_3 - 2w\eta_4 \in 4\mathcal{O}_K$ we have $z \equiv w \equiv 0 \pmod{2}$. On the

other hand, we have that

$$\begin{aligned}
\varepsilon_1 {}^{s_2}\alpha - \alpha &= [x(u_1 - 1) - yv_1m_1] \eta_1 \\
&\quad + [xv_1 - y(u_1 + 1)] \eta_2 \\
&\quad + \left[z(u_1 - 1 - v_1d) + w \left\{ u_1 - v_1 \left(\frac{m_1/d + d}{2} \right) \right\} \right] \eta_3 \\
&\quad + [2zv_1d + w(-u_1 - 1 + v_1d)] \eta_4 \\
&\in 4\mathcal{O}_K
\end{aligned}$$

where $\left(\frac{m_1/d+d}{2}\right)$ is an integer, which yields the system of congruence equations

$$(7.21) \quad x(u_1 - 1) + yv_1 \equiv 0 \pmod{4}$$

$$(7.22) \quad xv_1 - y(u_1 + 1) \equiv 0 \pmod{4}$$

$$(7.23) \quad wu_1 \equiv 0 \pmod{4}$$

since $m_1 \equiv 3 \pmod{4}$, $u_1 \not\equiv v_1 \pmod{2}$, $z \equiv w \equiv 0 \pmod{2}$, $d \equiv 1 \pmod{2}$, and $\frac{m_1/d+d}{2} \equiv 0 \pmod{2}$.

- (i) Suppose v_1 is odd and so u_1 is even. Then equations (7.21) and (7.22) are equivalent and equation (7.23) is trivial. We obtain $x \equiv y \pmod{4}$ if $u_1 + 1 \equiv v_1 \pmod{4}$ and we obtain $x \equiv -y \pmod{4}$ if $u_1 - 1 \equiv v_1 \pmod{4}$. Together with the condition $z \equiv w \equiv 0 \pmod{2}$, we have that $[Z_{c_1}(4\mathcal{O}_K) : 4\mathcal{O}_K] = 16$. Hence $|H^1(G, \mathcal{O}_K)_{c_1}| = 4$, which is the case that $|\widehat{H}^0(G, \mathcal{O}_K)_c| \neq |H^1(G, \mathcal{O}_K)_c|$.
- (ii) Suppose v_1 is even. Note that $v_1 \equiv 0 \pmod{4}$. Indeed, since u_1 is odd, we have $\left(\frac{u_1-1}{2}\right)\left(\frac{u_1+1}{2}\right) = m_1\left(\frac{v_1}{2}\right)^2$ with integer terms. As the multiple of consecutive numbers, the left hand side is even, so $v_1/2$ is even. Now, equation (7.23) gives $w \equiv 0 \pmod{4}$. Equations (7.21) and (7.22) gives that if $u_1 \equiv 1 \pmod{4}$ then $y \equiv 0 \pmod{2}$ and if $u_1 \equiv 3 \pmod{4}$ then $x \equiv 0 \pmod{2}$. Together with the conditions $z \equiv 0 \pmod{2}$ and $w \equiv 0 \pmod{4}$, we have that $[Z_{c_1}(4\mathcal{O}_K) : 4\mathcal{O}_K] = 16$ and $|H^1(G, \mathcal{O}_K)_{c_1}| = 4$.

(b) $c = c_2$.

Let $\alpha = x\eta_1 + y\eta_2 + z\eta_3 + w\eta_4 \in Z_{c_2}(4\mathcal{O}_K)$. Since ${}^{s_2}\alpha - \alpha = -2y\eta_2 + w\eta_3 - 2w\eta_4 \in 4\mathcal{O}_K$, we have $y \equiv 0 \pmod{2}$ and $w \equiv 0 \pmod{4}$.

Now from the condition

$$\begin{aligned}
\varepsilon_2 {}^{s_1}\alpha - \alpha &= \left[x(u_2 - 1) - zv_2m_2 - wv_2 \left(\frac{m_2}{2} \right) \right] \eta_1 \\
&\quad + \left[y(u_2 - 1) - wv_2 \left(\frac{m_2}{2d} \right) \right] \eta_2 \\
&\quad + [xv_2 - yv_2d - z(u_2 + 1)] \eta_3 \\
&\quad + [2yv_2d - w(u_2 + 1)] \eta_4 \\
&\in 4\mathcal{O}_K
\end{aligned}$$

we obtain the system of congruence equations

$$(7.24) \quad x(u_2 - 1) \equiv 0 \pmod{4}$$

$$(7.25) \quad xv_2 - z(u_2 + 1) \equiv 0 \pmod{4}$$

since $m_2 \equiv 2 \pmod{4}$, $u_2 \equiv 1 \pmod{2}$, and $v_2 \equiv 0 \pmod{2}$. If $u_2 \equiv 3 \pmod{4}$, the system of congruence equations turns to $x \equiv 0 \pmod{2}$. When $u_2 \equiv 1 \pmod{4}$, if $v_2 \equiv 0 \pmod{4}$ then the system of equations implies that $z \equiv 0 \pmod{2}$, and if $v_2 \equiv 2 \pmod{4}$, one does that $x \equiv z \pmod{2}$. Therefore, we have $Z_{c_2}(4\mathcal{O}_K)$ is the set of elements $x\eta_1 + y\eta_2 + z\eta_3 + w\eta_4$ with $x \equiv y \equiv 0 \pmod{2}$ and $w \equiv 0 \pmod{4}$ if $u_2 \equiv 3 \pmod{4}$, the set of elements $x\eta_1 + y\eta_2 + z\eta_3 + w\eta_4$ with $y \equiv z \equiv 0 \pmod{2}$ and $w \equiv 0 \pmod{4}$ if $u_2 \equiv 1 \pmod{4}$ and $v_2 \equiv 0 \pmod{4}$, and the set of elements defined by the conditions $x \equiv z \pmod{2}$, $y \equiv 0 \pmod{2}$, and $w \equiv 0 \pmod{4}$ if $u_2 \equiv 1 \pmod{4}$ and $v_2 \equiv 2 \pmod{4}$. In any cases the index of $Z_{c_2}(4\mathcal{O}_K)$ from $4\mathcal{O}_K$ is 16 and we have $|H^1(G, \mathcal{O}_K)_{c_2}| = 4$. Comparing to the order of 0-Tate twisted cohomology, we have $|\widehat{H}^0(G, \mathcal{O}_K)_{c_2}| \neq |H^1(G, \mathcal{O}_K)_{c_2}|$ if $u_2 \equiv 3 \pmod{4}$, or if $u_2 \equiv 1 \pmod{4}$ and $v_2 \equiv 2 \pmod{4}$.

(c) $c = c_3$.

Let $\alpha = x\eta_1 + y\eta_2 + z\eta_3 + w\eta_4 \in Z_{c_3}(4\mathcal{O}_K)$. Since ${}^{s_3}\alpha - \alpha = -2y\eta_2 - (w + 2z)\eta_3 \in 4\mathcal{O}_K$, we have $y \equiv 0 \pmod{2}$ and $w + 2z \equiv 0 \pmod{4}$. The latter condition shows that w is even, and that $w \equiv 0 \pmod{4}$ if and only if $z \equiv 0 \pmod{2}$.

Now from the condition

$$\begin{aligned} \varepsilon_3^{s_1}\alpha - \alpha &= \left[x(u_3 - 1) - wv_3 \left(\frac{m_3}{2} \right) \right] \eta_1 \\ &+ \left[y(u_3 - 1) - zv_3 \left(\frac{m_2}{d} \right) - wv_3 \left(\frac{m_2}{2d} \right) \right] \eta_2 \\ &+ \left[-xv_3 + yv_3 \left(\frac{m_1}{d} \right) - z(u_3 + 1) \right] \eta_3 \\ &+ [2xv_3 - w(u_3 + 1)] \eta_4 \\ &\in 4\mathcal{O}_K, \end{aligned}$$

we obtain the system of congruence equations

$$(7.26) \quad x(u_3 - 1) \equiv 0 \pmod{4}$$

$$(7.27) \quad -xv_3 - z(u_3 + 1) \equiv 0 \pmod{4},$$

and it turns to be following equations due to u_3, v_3 modulo 4: $z \equiv 0 \pmod{2}$ and $w \equiv 0 \pmod{4}$ if $u_3 \equiv 1 \pmod{4}$ and $v_3 \equiv 0 \pmod{4}$; $x \equiv z \pmod{2}$ if $u_3 \equiv 1 \pmod{4}$ and $v_3 \equiv 2 \pmod{4}$; and $x \equiv 0 \pmod{2}$ if $u_3 \equiv 3 \pmod{4}$.

Hence $Z_{c_3}(4\mathcal{O}_K)$ is the set of elements $\alpha = x\eta_1 + y\eta_2 + z\eta_3 + w\eta_4$ satisfying

$$\begin{cases} \left. \begin{array}{l} y \equiv z \equiv 0 \pmod{2} \\ w \equiv 0 \pmod{4} \end{array} \right\} & \text{if } u_3 \equiv 1 \pmod{4} \text{ and } v_3 \equiv 0 \pmod{4} \\ \left. \begin{array}{l} x \equiv z \pmod{2} \\ y \equiv 0 \pmod{2} \\ w \equiv 2z \pmod{4} \end{array} \right\} & \text{if } u_3 \equiv 1 \pmod{4} \text{ and } v_3 \equiv 2 \pmod{4} \\ \left. \begin{array}{l} x \equiv y \equiv 0 \pmod{2} \\ w \equiv 2z \pmod{4} \end{array} \right\} & \text{if } u_3 \equiv 3 \pmod{4} \end{cases}$$

and we have that $[Z_{c_3}(4\mathcal{O}_K) : 4\mathcal{O}_K] = 16$ and $|H^1(G, \mathcal{O}_K)_{c_3}| = 4$ in any cases. Also we have $|\widehat{H}^0(G, \mathcal{O}_K)_{c_3}| \neq |H^1(G, \mathcal{O}_K)_{c_3}|$ if $u_3 \equiv 3 \pmod{4}$ or if $u_3 \equiv 1, v_3 \equiv 2 \pmod{4}$.

REMARK 7.2. It is easy to see that $|H^1(G, \mathcal{O}_K)| = |\widehat{H}^1(G, \mathcal{O}_K)| = 2$ or 4 for Type II or Type III, respectively. As a summary, we have $|H^1(G, \mathcal{O}_K)_c| \neq |\widehat{H}^1(G, \mathcal{O}_K)_c|$ in Type III, when

- (a) $c = c_1$ and v_1 is odd
- (b) $c = c_i$ and $u_i \equiv 3 \pmod{4}$, for $i = 2, 3$
- (c) $c = c_i, u_i \equiv 1 \pmod{4}$, and $v_i \equiv 2 \pmod{4}$, for $i = 2, 3$.

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Department of Liberal Arts
College of Engineering
Hongik University
Seoul 04066, Republic of Korea
E-mail: sherwood@hongik.ac.kr